MATH 521 Definitions and Theorems

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- Def A **set** is a collection of objects called elements
- Def A is a **subset** of B ($A \subset B$ or $A \subseteq B$) if $\forall x \in A, x \in B$
- Def If A, B are sets, the **Cartesian product** $A \times B$ is the set of ordered pairs $\forall a \in A, \forall b \in B, (a, b)$
- Def A, B are sets; a **function** $F : A \to B$ is a subset $F \subset A \times B$ s.t. $(a, b) \in F \wedge (a, b') \in F \implies b = b'$; $\forall a \in A, \exists b \in B, (a, b) \in F$
- Def $f : A \to B$ is **injective** if $\forall a, a' \in A, f(a) = f(a') \implies a = a'$, i.e., if $\forall b \in B$ at most 1 $a \in A$ with $f(a) = b$
- Def $f : A \to B$ is **surjective** if $\forall b \in B, \exists a \in A, F(a) = b$, i.e., if $\forall b \in B$ at least 1 $a \in A$ with $f(a) = b$
- Def $f : A \to B$ is **bijective** iff f is both injective and surjective
- Fact When $f : A \to B$ is bijective, $\exists f^{-1} : B \to A$ s.t. $f^{-1}(b) = \text{the unique } a \in A \text{ s.t. } f(a) = b$
- Def Given $f : A \to B, g : B \to C$; the **function composition** is gf or $g \circ f : A \to C$ s.t. $\forall a \in A, g \circ f(a) = g(f(a))$
- Fact Target of f must match source of g that not all pairs of functions can be composed
- Fact Function composition is not commutative
- Def A **relation** from A to B is a subset $R \subset A \times B$
- Def An **ordering** on S is a relation $\leq$ on $S \times S$ s.t. (comparability) $\forall x, y \in S$ *exactly* one of $x < y$, $y < x$, $x = y$ is true; (transitivity) $\forall x, y, z \in S$ if $x \leq y$ and $y \leq z$, then $x \leq z$
- Def An **ordered set** is a set $(S, \leq)$ with an order $\leq$ on S
- Fact If $x \leq y$ and $y \leq x$, then $x = y$
- Def S is an ordered set, $E \subset S$; if $\exists s \in S$ s.t. $\forall e \in E, s \geq e$, then E is **bounded above** in S, and s is an **upper bound** for E
- Def s is a **least upper bound**, $\sup E$, or the **supremum** of E if s is an upper bound for E; t is an upper bound for $E \implies t \geq s$
- Def Similarly, **greatest lower bound** for E is called **infimum** or $\inf E$
- Def A **field** is a set where we can do arithmetic ($+, -, \times, \div$ except 0). We require the existence of $0, 1$ and the rules $0 + x = x$, $x + y = y + x$, $(x + y) + z = x + (y + z)$, $x(y + z) = xy + xz$, $xy = yx$
- Def An **ordered field** is a field F with an ordering $<$ s.t. for $x, y, z \in F$ with $y < z$, $x + y < x + z$; if $x, y \in F$, $x > 0, y > 0$, $xy > 0$
- Fact In an ordered field, $\forall x \in F\ x^2 \geq 0$
- Fact A bounded ordered set $e \subset S$ has at most one supremum
- Fact $\left\{x \in \mathbb{Q}_{>0} : x^2 < 2\right\}$ has no supremum

- <u>Def</u> A nonempty set S has the **least upper bound property** if every nonempty $E \subset S$ which is bounded above has a least upper bound
- <u>Thm</u> If S has the least upper bound property, then S has the greatest lower bound property
- <u>Def</u> An ordered field is **complete** if it has the least upper bound property
- <u>Rmk</u> $\mathbb{Q}$ is not complete, a finite ordered set has the least upper bound property
- <u>Thm</u> There is a complete ordered field $\mathbb{R}$
- <u>Def</u> If F, G are ordered fields, an **isomorphism** is a function $f : F \to G$ s.t. f is a bijection; $f(x) > f(y)$ iff $x > y$; $f(x+y) = f(x) + f(y)$; $f(xy) = f(x)f(y)$ (F, G being "the same field with the elements labelled differently")
- <u>Prop</u> $F = \mathbb{R}$ has **the Archimedean property**: for any $x, y > 0, \exists n \in \mathbb{Z}_{>0}$ s.t. $nx > y$
- <u>Prop</u> $F = \mathbb{R}, n \in F_{>0}$, then there is a unique $y \in F$ with $y^n = x$ ("n^{th} roots exist")
- <u>Lemma</u> a is an upper bound for $\{z : z \geq 0, z^n \leq x\}$ iff $a^n \geq x$
- <u>Def</u> A **Cauchy sequence of** $\mathbb{Q}$ is a sequence $\mathbf{a} = a_1, a_2, a_3, \ldots$ s.t. $\forall \epsilon > 0, \exists N \in \mathbb{Z}_{>0}$ s.t. $\forall i, j > N, |a_i - a_j| < \epsilon$
- <u>Def</u> An **equivalence relation** on S is a relation $\sim \in S \times S$ s.t. $\forall x, x \sim x$; $\forall x, y, x \sim y \iff y \sim x$; $\forall x, y, z, x \sim y \wedge y \sim z \implies x \sim z$ (which is a bijection)
- <u>Def</u> An equation relation $\sim$ partitions S into disjoint subsets (i.e. **equivalence classes**) s_i, each one of the form $\{s \in S : s \sim s_0\}$ for some s_0
- <u>Def</u> A **real number** is an equivalence class of Cauchy sequences under this relation
- <u>Prop</u> If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are Cauchy sequences, $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{b} \sim \mathbf{c}$, then $\mathbf{a} \sim \mathbf{c}$
- <u>Def</u> $\mathbf{a} > \mathbf{b}$ if $\exists \delta > 0, N$ s.t. $a_i > b_i + \delta$ for all $i > N$
- <u>Prop</u> If $x, y \in \mathbb{R}$, *exactly* one of $x < y, x > y, x = y$ is true
- <u>Def</u> A **Cauchy sequence of** $\mathbb{R}$ is a sequence $\mathbf{x} = x_1, x_2, \ldots x_i \in \mathbb{R}$ s.t. $\forall \epsilon > 0, \exists N$ s.t. $|x_i - y| < \epsilon$
- <u>Def</u> A sequence of reals $\mathbf{x}$ has **limit** y if $\forall \epsilon > 0, \exists N$ s.t. $\forall i > N, |x_i - y| < \epsilon$
- <u>Thm</u> $\mathbb{R}$ has the least upper bound property
- <u>Thm</u> (**Monotone Convergence Thm**) A non-decreasing sequence of $x_i \in \mathbb{R}$ which is bounded above is Cauchy and thus converges to a limit (everything that rises must converge)
- <u>Def</u> $\overline{R}$, the **extended reals**, is an ordered set $\mathbb{R} \cup \{-\infty, \infty\}$
- <u>Def</u> If $\mathbf{x} = x_1, x_2, \ldots$ is a sequence in $\overline{R}$, we say $\lim \mathbf{x} = \infty$ if $\forall E, \exists N, \forall i > N, x_i > E$
- <u>Fact</u> $\overline{R}$ is not a field
- <u>Def</u> The equivalence classes under $\sim$ are called **cardinalities**
- <u>Fact</u> The equivalence classes of finite sets are $\mathbb{N}$ (by pigeonhole principle)
- <u>Def</u> A set S is **infinite** if $\exists f : S \hookrightarrow S$ (monomorphism) an injection which is not a bijection
- <u>Fact</u> An infinite set cannot be equivalent to a finite set
- <u>Def</u> We say S is **countably infinite** if $S \sim \mathbb{Z}_{>0}$
- <u>Prop</u> $\mathbb{Z}$ is countable
- <u>Fact</u>
    - If S, T are countable, so is $S \cup T$
    - If S countable, any subset of S is countably infinite or finite
    - If S, T are countable, so is $S \times T$
- <u>Def</u> if S, T are sets, we denote by S^T the set of functions from T to S
- <u>Fact</u>

- If S, T are finite sets, $|S^T| = |S|^{|T|}$
- $\{0,1\}^T$ = set of subsets of T
- $S^\emptyset = \emptyset$

- Thm (**Cantor's Diagonal Argument**) There are uncountable sets, i.e., $\{0,1\}^{\mathbb{Z}_{\geq 0}}$
- Corollary $\mathbb{R}$ is uncountable
- Def A **vector space** over a field k (a k-vector space) is a set V, whose elements are called **vectors** with operations $+$ (addition) and $\cdot$ (scalar multiplication)
- Def V is a vector space over $\mathbb{R}$; a **norm** on V is a function $||\cdot|| : V \to \mathbb{R}$ s.t. (nonnegativity) $||v|| \geq 0 \forall v \in V$, and $||v|| = 0$ iff $v = \mathbf{0}$; (homogeneity) $||\lambda v|| = |\lambda| ||v|| \forall \lambda \in \mathbb{R}, v \in V$; (triangle inequality) $\forall v, w \in V, ||v + w|| \leq ||v|| + ||w||$
- Rmk the set of functions $f : \mathbb{R} \to \mathbb{R}$ bounded between -1 and 1 are not a vector space
- Def The **norm ball** $B_{v,||\cdot||}(a)$ is the set $\{v \in V : ||v|| \leq a\}$
- Def A subset S of a vector space V is **convex** if $\forall v, w \in S$, the line segment $\overline{vw}$ is continued in S
- Fact Norm balls are always convex
- Def
  - (L^p **norm**) $||(x_1, \ldots, x_n)||_p = (|x_1|^p + \cdots + |x_n|^p)^{\frac{1}{p}}$
  - (L^∞ **norm**) $||(x_1, \ldots, x_n)||_\infty = \max_i |x_i|$
  - (**sup norm**)$||f||_{\text{sup}} = \sup_x |f(x)|$
- Fact $||v||_1 \geq ||v||_2 \geq ||v||_\infty, ||v||_1 \leq 2||v||_\infty, ||v||_1 \leq \sqrt{d}||v||_2$ for $v \in \mathbb{R}^d$
- Def An **inner product space** is a vector space V with a function $<,>: V \times V \to \mathbb{R}$ s.t. (symmetry) $\langle x, y \rangle = \langle y, x \rangle$; $\langle x_1 + x_2, y \rangle = \langle x_1, y \rangle + \langle x_2, y \rangle$; (linearity in first component) $\langle \lambda x + y, z \rangle = \lambda \langle x, z \rangle + \langle y, z \rangle$; (positive-definiteness) $\langle x, x \rangle \geq 0$ and $\langle x, x \rangle = 0$ iff $x = 0$
- Thm For every inner product space V, $||v|| := \langle v, v \rangle^{\frac{1}{2}}$ is a norm, i.e., every inner product space is a normed space
- Thm (**Cauchy-Schwarz Inequality**) $\langle x, y \rangle \leq ||x|| ||y||$
- Corollary If $v, w \in \mathbb{R}^n$, Cauchy-Schwarz says $\cos\theta = \frac{\langle v, w \rangle}{||v|| ||w||} \leq 1$, where θ is the angle between v and w
- Fact correlation coefficient ρ measures the relation between the variables, i.e., the angle between $\vec{x}$ and $\vec{y}$ if everything is normalized to the mean 0
  - $\rho = 1$ iff $\forall i, a > 0, y_i = ax_i$ iff $\theta = 0$
  - $\rho > 0$ iff $\vec{x}, \vec{y}$ are at an acute angle; $\rho < 0$ for obtuse angle; $\rho = 0$ for orthogonal/perpendicular
- Def A **metric space** is a set X with function $d : X \times X \to \mathbb{R}$ s.t. (nonnegativity) $d(x, y) \geq 0$, with $d(x, y) = 0$ iff $x = y$; (symmetry) $d(x, y) = d(y, x)$; (triangle inequality) $d(x, z) \leq d(x, y) + d(y, z)$
- Fact If V is a normed vector space, then the function $d(x, y) = ||x - y||$ is a metric
- Def X is metric space with metric d; the **open ball** of radius r around $x \in X$ is $B_{X,d}(x, r) = \{y \in X, d(x, y) < r\}$
- Def X is a metric space, $S \subset X$; the **interior** of S with respect to X, denoted $\text{int}_X(S)$, is the set $\{s \in S : \exists \epsilon > 0, B_X(s, \epsilon) \subset S\}$
- Def A subset $S \subset X$ is **open** if $\text{int}_X(S) = S$
- Thm $\forall S \subset X$, if $\text{int}(\text{int}(S)) = \text{int}(S)$, then $\text{int}(S)$ is open
- Prop The union of any collection of open/closed sets is the same (open/closed)
- PropThe intersection of any finite collection of open/closed sets is the same (open/closed)

- Prop Every open set $S \subset X$ is a union of some collection of open balls
- Def X is a metric space, $p \in X$; a **neighborhood** of p is an open set $U \ni p$
- Def E is a subset of metric space X; $x \in X$ is an **accumulation point** or **limit point** of E in X (i.e., $x \in L_X(E)$) if for every neighborhood $U \ni x$, $U \cap E$ contains a point not equal to x, i.e., $\forall B(x,\epsilon)\epsilon > 0$
- Def A point x is an **isolated point** of E in X if $x \in E$ and x is not a limit point of E in X
- Prop If $X = \mathbb{R}^n$ with Euclidean metric and $E \subset X$, then any point in $\text{int}(E)$ is an accumulation point for E
- Def Let $x_1, x_2, \ldots$ be a sequence of points in a metric space X; $\lim_{i\to\infty} x_i = x$ if, for every neighborhood U of x, $\exists N_U$ s.t. $\forall i > N_U$, $x_i \in U$; i.e.; $\forall \epsilon > 0, \exists N_\epsilon$ s.t. $\forall i > N_\epsilon$, $x_i \in B(x,\epsilon)$ i.e. $d(x, x_i) < \epsilon$
- Thm Let $E \subset X$; $x \in L_X(E)$ iff $\exists$ a sequence $x_1, x_2, \cdots \in E \setminus X$ with $\lim_{i\to\infty} x_i = x$
- Def E is **closed** in X if $L_X(E) \subset E$
- Def $E \subset X$ is **closed** if its complement $\overline{E}$ $(:= X \setminus E)$ is open
- Def The **closure** of E in X is $\text{clos}(E) = E \cup L_E(X)$; i.e.; $\text{clos}(E) = \overline{\text{int}(\overline{E})}$, or $\text{clos}(E) =$ intersection of all closed subsets of X containing E
- Rmk $\text{clos}(E) = E$ iff E is closed
- Def x is a **subsequential limit** of $x_1, x_2, \ldots$ if $\exists$ a sequence $x_{i_1}, x_{i_2}, \ldots$ $(i_1 < i_2 < \cdots)$ with $\lim_{j\to\infty} x_{i_j} = x$
- Prop If $\lim_{i\to\infty} x_i = x$, then x is the only subsequential limit
- Thm (**Bolzano–Weierstrass Thm**) Any bounded sequence of real numbers has a subsequential limit
- Thm (**Baby Bolzano–Weierstrass Thm**) If K is a finite set of real numbers, then any sequence $x_1, x_2, \cdots \in K$ has a subsequential limit (by pigeonhole principle)
- Prop Let $x_1, x_2, \ldots$ a sequence in X, $E = \{x_1, x_2, \ldots\} \subset X$; if x is an accumulation point of E, it is a subsequential limit of $x_1, x_2, \ldots$
- Def Let K be a subset of a metric space X; K is **compact** if, for any collection of open sets of X ($\{U_s\}_{s\in S}$) which **covers** K (i.e. $K \subset \cup_{s\in S} U_s$)
  - there is a finite subcollection $U_1, U_2, \ldots, U_N$ which still covers K
- Fact A finite set K is compact
- Fact Any unbounded subset of $\mathbb{R}$ (including $\mathbb{R}, \mathbb{Z}$) is noncompact
- Fact $(0,1)$ is noncompact; $[0,1]$ is compact
- Thm (**Heine-Borel Thm**) $\forall S \subset \mathbb{R}^n$, S is closed and bounded iff S is compact
- Thm Let $x_1, x_2, \ldots$ be an infinite sequence in a compact subset $K \subset X$; then $x_1, x_2, \ldots$ has a convergent subsequence where limit is in K