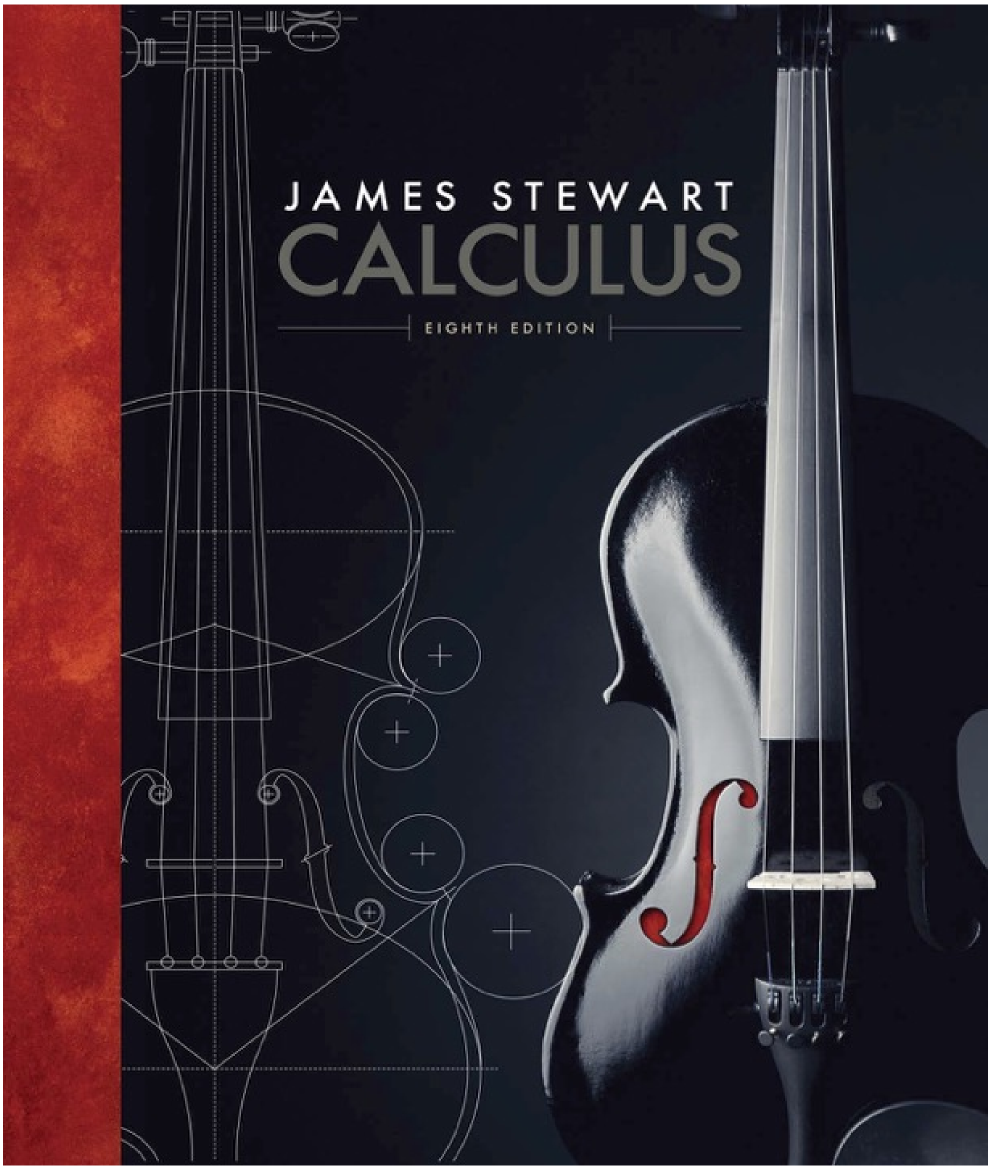
MATH 234:
Calculus–Functions of Several Variables

Chapter 12 - Chapter 16, Stewart Calculus 8th Edition

Updated 16 September 2021

- Vectors and Geometry of Space
- Vector Functions
- Partial Derivatives
- Multiple Integrals
- Vector Calculus



Vectors and Geometry of Space

- 3D Coordinate Systems
  - Distance
    - 2D
      - $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$
      - $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
    - 3D
      - $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$
      - $|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
  - Sphere
    - Center $C(h, k, l)$
    - $(x - h)^2 + (y - k)^2 + (z - l)^2 = r^2$
- Vectors
  - Calculation
    - Addition
    - Scalar Multiplication
  - Magnitude / Length
    - 2D
      - $\mathbf{a} = \langle a_1, a_2 \rangle$
      - $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2}$
    - 3D
      - $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$
      - $|\mathbf{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$
  - Properties
    - $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$
    - $\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}$
    - $\mathbf{a} + \mathbf{0} = \mathbf{a}$
    - $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$
    - $c(\mathbf{a} + \mathbf{b}) = c\mathbf{a} + c\mathbf{b}$
    - $(c + d)\mathbf{a} = c\mathbf{a} + d\mathbf{a}$
    - $(cd)\mathbf{a} = c(d\mathbf{a})$
    - $1\mathbf{a} = \mathbf{a}$
  - Standard Basis Vectors
    - $\mathbf{i} = \langle 1, 0, 0 \rangle$
    - $\mathbf{j} = \langle 0, 1, 0 \rangle$
    - $\mathbf{k} = \langle 0, 0, 1 \rangle$
- Dot Product
  - Definition
    - $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$
    - $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
  - Properties
    - $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
    - $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
    - $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

    - $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
  - Theorem
    - $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$
    - $\mathbf{a} \cdot \mathbf{b} = 0$, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal
  - Direction Angles
    - $\frac{\mathbf{a}}{|\mathbf{a}|} = \langle \cos\alpha, \cos\beta, \cos\gamma \rangle$
  - Projections
    - Scalar projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$
    - Vector projection of $\mathbf{b}$ onto $\mathbf{a}$: $\text{proj}_{\mathbf{a}} = \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$
- Cross Product
  - Definition: $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$
  - Theorem: $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$
  - Theorem: $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}| \sin\theta$ when $(0 \leqslant \theta \leqslant \pi)$
  - Corollary: $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$
  - Intepretation: Length of $\mathbf{a} \times \mathbf{b}$ = A of parallelogram
  - Properties
    - $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$
    - $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$
    - $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$
    - $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$
    - $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$
    - $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$
  - Volume of Parallelepiped: $V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$
- Equations of Lines and Planes
  - Line Segment 1
    - $\mathbf{v}$: direction
    - $\mathbf{r} = \mathbf{r}_0 + t\mathbf{v}$
  - **Line Segment 2**
    - starts at $\mathbf{r}_0$, ends at $\mathbf{r}_1$
    - $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1, 0 \leqslant t \leqslant 1$
  - Planes
    - Vector Equation of Plane: $\mathbf{n} \cdot \mathbf{r} = \mathbf{n} \cdot \mathbf{r}_0$
      - **Plane Contains Points $\mathbf{A}, \mathbf{B}, \mathbf{C}$**
        - $\overrightarrow{AB} = \mathbf{B} - \mathbf{A}$
        - $\overrightarrow{AC} = \mathbf{C} - \mathbf{A}$
        - $\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{BC}$
    - Scalar Equation of Plane
      - Through $P_0(x_0, y_0, z_0)$ with normal vector $\mathbf{n} = \langle a, b, c \rangle$
      - $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$
    - Linear Equation: $ax + by + cz + d = 0$ where $d = -(ax_0 + by_0 + cz_0)$
  - Distance
    - $\mathbf{b}$: vector corresponding to $\overrightarrow{P_0P_1}$
    - $D = |\text{comp}_{\mathbf{n}}\mathbf{b}| = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$

- Cylinders and Quadric Surfaces
  - Cylinders
  - Quadric Surfaces
    - Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ (sphere when $a=b=c$)
    - Cone: $\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
    - Elliptic Paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
    - Hyperboloid of One Sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
    - Hyperbolic Paraboloid: $\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$
    - Hyperboloid of Two Sheets: $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Vector Functions

- Vector Functions and Space Curves
  - Limits
    - $\mathbf{r}(t)=\langle f(t),g(t),h(t)\rangle$
    - $\lim_{t\to a}\mathbf{r}(t)=\left\langle \lim_{t\to a}f(t),\lim_{t\to a}g(t),\lim_{t\to a}h(t)\right\rangle$
  - Space Curves: $x=f(t)$, $y=g(t)$, $z=h(t)$
- Derivatives and Integrals of Vector Functions
  - $\frac{d\mathbf{r}}{dt}=\mathbf{r}'(t)=\lim_{h\to 0}\frac{\mathbf{r}(t+h)-\mathbf{r}(t)}{h}$
  - **Unit Tangent Vector**: $\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$
  - $\mathbf{r}'(t)=\langle f'(t),g'(t),h'(t)\rangle=f'(t)\mathbf{i}+g'(t)\mathbf{j}+h'(t)\mathbf{k}$
  - Differential Rules
    - $\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)]=\mathbf{u}'(t)+\mathbf{v}'(t)$
    - $\frac{d}{dt}[c\mathbf{u}(t)]=c\mathbf{u}'(t)$
    - $\frac{d}{dt}[f(t)\mathbf{u}(t)]=f'(t)\mathbf{u}(t)+f(t)\mathbf{u}'(t)$
    - $\frac{d}{dt}[\mathbf{u}(t)\cdot\mathbf{v}(t)]=\mathbf{u}'(t)\cdot\mathbf{v}(t)+\mathbf{u}(t)\cdot\mathbf{v}'(t)$
    - $\frac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)]=\mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$
    - $\frac{d}{dt}[\mathbf{u}(f(t))]=f'(t)\mathbf{u}'(f(t))$
  - Integrals
    - $\int_a^b\mathbf{r}(t)\,dt=\lim_{n\to\infty}\sum_{i=1}^{n}\mathbf{r}(t_i^*)\,\Delta t=\lim_{n\to\infty}\left[\left(\sum_{i=1}^{n}f(t_i^*)\,\Delta t\right)\mathbf{i}+\left(\sum_{i=1}^{n}g(t_i^*)\,\Delta t\right)\mathbf{j}+\left(\sum_{i=1}^{n}h(t_i^*)\,\Delta t\right)\mathbf{k}\right]$
    - $\int_a^b\mathbf{r}(t)\,dt=\left(\int_a^b f(t)\,dt\right)\mathbf{i}+\left(\int_a^b g(t)\,dt\right)\mathbf{j}+\left(\int_a^b h(t)\,dt\right)\mathbf{k}$
- Arc Length and Curvature
  - Length of Curve
    - 2D: $L=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2}\,dt$
    - 3D: $L=\int_a^b\sqrt{[f'(t)]^2+[g'(t)]^2+[h'(t)]^2}\,dt$
    - $L=\int_a^b|\mathbf{r}'(t)|\,dt$
    - Arc Length Function: $\frac{ds}{dt}=|\mathbf{r}'(t)|$
  - Curvature
    - $\kappa=\left|\frac{d\mathbf{T}}{ds}\right|=\frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}=\frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}=\frac{|f''(x)|}{[1+(f'(x))^2]^{3/2}}$
    - **Unit Normal Vector**: $\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$
    - **Binormal Vector**: $\mathbf{B}(t)=\mathbf{T}(t)\times\mathbf{N}(t)$

- Motion in Space: Velocity and Acceleration
  - Velocity: $\mathbf{v}(t) = \mathbf{r}'(t)$
  - Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$
  - Newton's Second Law: $\mathbf{F}(t) = m\mathbf{a}(t)$
  - Projectile Motion: $x = (v_0 \cos\alpha)t, y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2$
  - Tangential and Normal Components of Acceleration
    - $v = |\mathbf{v}|$
    - $\mathbf{a} = v'\mathbf{T} + \kappa v^2\mathbf{N}$
  - Law of Gravitation
    - $r = |\mathbf{r}|, \mathbf{u} = \frac{\mathbf{r}}{r}$
    - $\mathbf{F} = -\frac{GMm}{r^3}\mathbf{r} = -\frac{GMm}{r^2}\mathbf{u}$

Partial Derivatives

- Limit: $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x}) = L$
  - Definition
    - $\mathbf{x} = \langle x, y\rangle$
    - $\mathbf{a} = \langle a, b\rangle$
    - $|\mathbf{x} - \mathbf{a}| = \sqrt{(x-a)^2 + (y-b)^2}$
  - If $0 < |\mathbf{x} - \mathbf{a}| < \delta, |f(\mathbf{x}) - L| < \epsilon$
  - If $L_1 \neq L_2$ on paths C_1 and C_2, $\lim\limits_{\mathbf{x}\to\mathbf{a}} f(\mathbf{x})$ DNE
- Derivative
  - Basic
    - $f_x(x, y) = \lim\limits_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$
    - $f_y(x, y) = \lim\limits_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}$
  - Notation
    - $f_x(x,y) = f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} f(x,y) = \frac{\partial z}{\partial x} = f_1 = D_1 f = D_x f$
    - $f_y(x,y) = f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} f(x,y) = \frac{\partial z}{\partial y} = f_2 = D_2 f = D_y f$
  - Clairaut's Theorem: $f_{xy}(a,b) = f_{yx}(a,b)$
  - Laplace's Equation (Harmonic Functions): $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$
- Tangent Planes and Linear Approximations
  - Tangent Plane to $z = f(x,y)$ at $P(x_0, y_0, z_0)$: $z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$
  - **Linearization**
    - $f(x,y) \approx f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)$
    - $\Delta f \approx \nabla f(\mathbf{x}) \cdot \Delta\mathbf{x}$
  - $z = f(x,y)$, then f differentiable if $\Delta z = f_x(a,b)\ \Delta x + f_y(a,b)\ \Delta y + \epsilon_1\ \Delta x + \epsilon_2\ \Delta y$
    - $\epsilon_1, \epsilon_2 \to 0$
    - $(\Delta x, \Delta y) \to (0,0)$
  - If f_x and f_y exist near (a,b) and are continuous at (a,b), then f is differentiable at (a,b)
  - Total Differential of $z = f(x,y)$: $dz = f_x(x,y)\ dx + f_y(x,y)\ dy = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$
- Chain Rule
  - Case 1
    - $z = f(x,y), x = g(t), y = h(t)$

        - $\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$
    - Case 2
        - $z = f(x, y), x = g(s, t), y = h(s, t)$
        - $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$
        - $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}$
    - General
        - $u = f(x_1, x_2, \dots, x_n), x_j = g(t_1, t_2, \dots, t_m), i = 1, 2, \dots, m$
        - $\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}$
    - Implicit Differentiation
        - $\frac{dy}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$
        - $\frac{dz}{dx} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{F_x}{F_z}$
        - $\frac{dz}{dy} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{F_y}{F_z}$
- Directional Derivatives and Gradient Vector
    - Directional Derivative
        - Definition
            - f at (x_0, y_0), direction of unit vector $\mathbf{u} = \langle a, b\rangle$
            - $D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb) - f(x_0, y_0)}{h}$
            - $D_{\mathbf{u}}f(x_0, y_0, z_0) = \lim_{h\to 0}\frac{f(x_0+ha, y_0+hb, z_0+hc) - f(x_0, y_0, z_0)}{h}$
            - $D_{\mathbf{u}}f(\mathbf{x}_0) = \lim_{h\to 0}\frac{f(\mathbf{x}_0+h\mathbf{u}) - f(\mathbf{x}_0)}{h}$
        - Theorem
            - $f = (x, y)$, direction of any unit vector $\mathbf{u} = \langle a, b\rangle$
            - $D_{\mathbf{u}}f(x, y) = f_x(x, y)a + f_y(x, y)b = \nabla f(x, y) \cdot \mathbf{u}$
            - $D_{\mathbf{u}}f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$
    - Gradient
        - Definition
            - $f(x, y)$, gradient ∇
            - $\nabla f(x, y) = \langle f_x(x, y), f_y(x, y)\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}$
            - $\nabla f = \langle f_x, f_y, f_z\rangle = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$
        - Tangent Plane
            - level surface $F(x, y, z) = k$ at $P(x_0, y_0, z_0)$
            - $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$
        - Normal Line
            - $\frac{x - x_0}{F_x(x_0, y_0, z_0)} = \frac{y - y_0}{F_y(x_0, y_0, z_0)} = \frac{z - z_0}{F_z(x_0, y_0, z_0)}$
- Maximum and Minimum Values
    - Critical Points / First Derivatives
        - $f_x(a, b) = 0$
        - $f_y(a, b) = 0$
    - Second Derivatives Test
        - $D = D(a, b) = f_{xx}(a, b)f_{yy}(a, b) - [f_{xy}(a, b)]^2 = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}$
        - Conditions
            - Local Minimum: $D > 0$ and $f_{xx}(a, b) > 0$
            - Local Maximum: $D > 0$ and $f_{xx}(a, b) < 0$

        - Saddle Point: $D < 0$
    - Absolute Extremes
        - f on closed, bounded set D
            1. critical points in D
            2. boundary of D
- **Lagrange Multipliers**
    - Definition: $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$
    - Method: find extreme values of $f(x, y, z)$ in $g(x, y, z) = k$
        1. Solve for x, y, z, and λ:
        $\begin{cases} \nabla f(x, y, z) = \lambda \nabla g(x, y, z) \\ g(x, y, z) = k \end{cases}$
        2. Evaluate f at all (x, y, z)
    - Two Constraints
        - $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)$

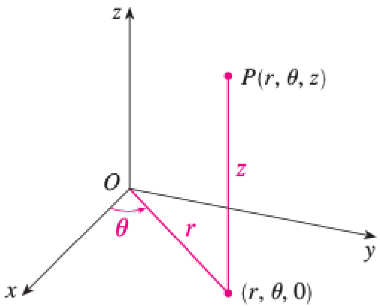
Multiple Integrals

- Double, Rectangles
    - $\iint\limits_R f(x, y)\, dA = \lim\limits_{m,n \to \infty} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A$
    - $V = \iint\limits_R f(x, y)\, dA$
    - Midpoint Rule
        - $\bar{x}_i$: midpoint of $[x_{i-1}, x_i]$
        - $\bar{y}_j$: midpoint of $[y_{j-1}, y_j]$
        - $\iint\limits_R f(x, y)\, dA \approx \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{n} f(\bar{x}_i, \bar{y}_j)\, \Delta A$
    - Fubini's Theorem
        - f continuous on $R = \{(x, y) \mid a \leqslant x \leqslant b,\ c \leqslant y \leqslant d\}$
        - $\iint\limits_R f(x, y)\, dA = \int_a^b \int_c^d f(x, y)\, dy\, dx = \int_c^d \int_a^b f(x, y)\, dx\, dy$
        - $\iint\limits_R g(x)h(y)\, dA = \int_a^b g(x)\, dx \int_c^d h(y)\, dy$ where $R = [a, b] \times [c, d]$
    - Average Value: $f_{\text{ave}} = \frac{1}{A(R)} \iint\limits_R f(x, y)\, dA$
- Double, General Regions
    - Type I
        - $D = \{(x, y) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x)\}$
        - $\iint\limits_D f(x, y)\, dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x, y)\, dy\, dx$
    - Type II
        - $D = \{(x, y) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y)\}$
        - $\iint\limits_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx\, dy$
    - Properties
        - $\iint\limits_D [f(x, y) + g(x, y)]\, dA = \iint\limits_D f(x, y)\, dA + \iint\limits_D g(x, y)\, dA$
        - $\iint\limits_D cf(x, y)\, dA = c \iint\limits_D f(x, y)\, dA$ where c is a constant

            - $\iint\limits_D f(x,y)\,dA \geqslant \iint\limits_D g(x,y)\,dA$ if $\forall (x,y) : f(x,y) \geqslant g(x,y)$
            - $\iint\limits_D 1\,dA = A(D)$
            - $mA(D) \leqslant \iint\limits_D f(x,y)\,dA \leqslant MA(D)$ if $\forall (x,y)$ in $D : m \leqslant f(x,y) \leqslant M$
- Double, Polar Coordinates
    - Polar Coordinates
        - $r^2 = x^2 + y^2$
        - $x = r\cos\theta$
        - $y = r\sin\theta$
        - $R = \{(r,\theta) \mid a \leqslant r \leqslant b,\ \alpha \leqslant \theta \leqslant \beta\}$
    - Double Integral
        - $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*)\ \Delta A_i = \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(r_i^*\cos\theta_j^*, r_i^*\sin\theta_j^*) r_i^*\ \Delta r\ \Delta\theta$
        - $\iint\limits_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\ r\ dr\ d\theta$
- Double, Application
    - Mass: $m = \iint\limits_D \rho(x,y)\,dA$
    - Moment about x-axis: $M_x = \iint\limits_D y\ \rho(x,y)\,dA$
    - Moment about y-axis: $M_y = \iint\limits_D x\ \rho(x,y)\,dA$
    - Center of Mass $(\bar{x}, \bar{y})$
        - $\bar{x} = \frac{M_y}{m}$
        - $\bar{y} = \frac{M_x}{m}$
    - Moment of Inertia / Second Moment
        - $I_x = \iint\limits_D y^2\ \rho(x,y)\,dA$
        - $I_y = \iint\limits_D x^2\ \rho(x,y)\,dA$
        - $I_0 = I_x + I_y = \iint\limits_D (x^2+y^2)\ \rho(x,y)\,dA$
    - Radius of Gyration of a Lamina about an axis
        - $mR^2 = I$
        - $m\bar{\bar{y}}^2 = I_x$
        - $m\bar{\bar{x}}^2 = I_y$
    - Probability
        - Joint Density Function: $P((X,Y) \in D) = \iint\limits_D f(x,y)\,dA$
        - Expected Values
            - X-mean: $\mu_1 = \iint\limits_{\mathbb{R}^2} x\ f(x,y)\,dA$
            - Y-mean: $\mu_2 = \iint\limits_{\mathbb{R}^2} y\ f(x,y)\,dA$
- Surface Area
    - $z = f(x,y), (x,y) \in D$
    - $A(S) = \iint\limits_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA = \iint\limits_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2}\,dA$
- Triple

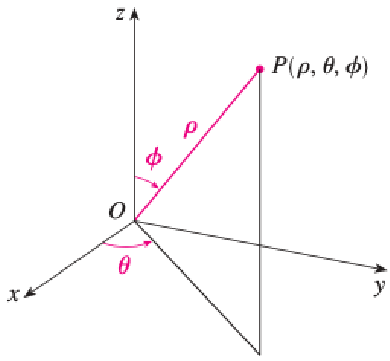
- Definition: $\iiint\limits_B f(x,y,z)\,dV = \lim\limits_{l,m,n\to\infty} \sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$
- Fubini's Theorem
  - f continuous on $B = [a,b] \times [c,d] \times [r,s]$
  - $\iiint\limits_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz = \int_a^b \int_r^s \int_c^d f(x,y,z)\,dy\,dz\,dx$ (any order)
- Type I
  - $E = \{(x,y,z) \mid a \leqslant x \leqslant b,\ g_1(x) \leqslant y \leqslant g_2(x),\ u_1) \mid\ \leqslant z \leqslant u_2(x,y)\}$
  - $\iiint\limits_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$
- Type II
  - $E = \{(x,y,z) \mid c \leqslant y \leqslant d,\ h_1(y) \leqslant x \leqslant h_2(y),\ u_1(x,y) \leqslant z \leqslant u_2(x,y)\}$
  - $\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$
- Type III
  - $E = \{(x,y,z) \mid (x,z) \in D,\ u_1(x,z) \leqslant y \leqslant u_2(x,z)\}$
  - $\iiint\limits_E f(x,y,z)\,dV = \iint\limits_D \left[\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy\right]\,dA$
- Application
  - $V(E) = \iiint\limits_E dV$
  - $\iiint\limits_E 1\,dV = \iint\limits_D \left[\int_{u_1(x,y)}^{u_2(x,y)} dz\right]\,dA = \iint\limits_D [u_2(x,y) - u_1(x,y)]\,dA$
  - Mass: $m = \iiint\limits_E \rho(x,y,z)\,dV$
  - Moments
    - $M_{yz} = \iiint\limits_E x\,\rho(x,y,z)\,dV$
    - $M_{xz} = \iiint\limits_E y\,\rho(x,y,z)\,dV$
    - $M_{xy} = \iiint\limits_E z\,\rho(x,y,z)\,dV$
  - Center of Mass at $(\bar{x}, \bar{y}, \bar{z})$ (Centeroid if m constant)
    - $\bar{x} = \frac{M_{yz}}{m}$
    - $\bar{y} = \frac{M_{xz}}{m}$
    - $\bar{z} = \frac{M_{xy}}{m}$
  - Moment of Inertia
    - $I_x = \iiint\limits_E (y^2 + z^2)\,\rho(x,y,z)\,dV$
    - $I_y = \iiint\limits_E (x^2 + z^2)\,\rho(x,y,z)\,dV$
    - $I_z = \iiint\limits_E (x^2 + y^2)\,\rho(x,y,z)\,dV$
  - Electric Charge: $Q = \iiint\limits_E \sigma(x,y,z)\,dV$
  - Joint Density Function: $P((X,Y,Z) \in E) = \iiint\limits_E f(x,y,z)\,dV$

- Triple, Cylindrical Coordinates
  - Cylindrical Coordinates
    -



- $x = r\cos\theta,\, y = r\sin\theta,\, z = z$
- $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$

- Formula: $\iiint\limits_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos\theta,\, r\sin\theta)}^{u_2(r\cos\theta,\, r\sin\theta)} f(r\cos\theta, r\sin\theta, z)\; r\; dz\; dr\; d\theta$

- Triple, Spherical Coordinates
  - Spherical Coordinates
    - 

    - $\rho \geqslant 0, 0 \leqslant \phi \leqslant \pi$
    - $x = \rho\sin\phi\cos\theta,\, y = \rho\sin\phi\sin\theta,\, z = \rho\cos\phi$
    - $\rho^2 = x^2 + y^2 + z^2$
  - Formula
    - $E = \{(\rho,\theta,\phi) \mid a \leqslant \rho \leqslant b,\ \alpha \leqslant \theta \leqslant \beta,\ c \leqslant \phi \leqslant d\}$
    - $\iiint\limits_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\; \rho^2\sin\phi\; d\rho\; d\theta\; d\phi$
- Multiple, Change of Variables
  - Double
    - $x = g(u,v), y = h(u,v)$
    - Jacobian: $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$
    - Change of Variables: $\iint\limits_R f(x,y)\,dA = \iint\limits_S f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\; du\; dv$
  - Triple
    - $x = g(u,v,w), y = h(u,v,w), z = k(u,v,w)$
    - Jacobian: $\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$
    - Change of Variables: $\iiint\limits_R f(x,y,z)\,dV =$
      $\iiint\limits_S f(x(u,v,w), y(u,v,w), z(u,v,w)) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right|\; du\; dv\; dw$

Vector Calculus

- Vector Fields
  - 2D vector: D a set in $\mathbb{R}^2$, vector field on $\mathbb{R}^2$ in D is $\mathbf{F}(x, y)$
    - $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$, P and Q are scalar functions / scalar fields
  - 3D vector: E a subset of $\mathbb{R}^3$, vector field on $\mathbb{R}^3$ in E is $\mathbf{F}(x, y, z) = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$
  - Gradient Vector Fields
    - $\nabla f(x, y) = f_x(x, y)\mathbf{i} + f_y(x, y)\mathbf{j}$
    - $\nabla f(x, y, z) = f_x(x, y, z)\mathbf{i} + f_y(x, y, z)\mathbf{j} + f_z(x, y, z)\mathbf{k}$
  - Conservative Vector Field
    - $\mathbf{F}$: there exists a function f that $\mathbf{F} = \nabla f$
    - f: potential function of $\mathbf{F}$
- Line Integrals
  - Definition
    - $\int_C f(x, y)\,ds = \lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$
    - $\int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2}\,dt$
    - $\int_C f(x, y, z)\,ds = \int_a^b f(x(t), y(t), z(t))\sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\,dt$
  - Line Integral to Arc Length
    - $\int_C f(x, y)\,dx = \int_a^b f(x(t), y(t))\,x'(t)\,dt$
    - $\int_C f(x, y)\,dy = \int_a^b f(x(t), y(t))\,y'(t)\,dt$
    - Line Segment
      - starts at $\mathbf{r}_0$, ends at $\mathbf{r}_1$
      - $\mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, $0 \leqslant t \leqslant 1$
  - Line Integral of Vector Fields
    - $\mathbf{F}$ is continuous vector field on smooth curve C given by vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$
    - $\mathbf{T}$ is unit tangent vector
    - $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\,dt = \int_C \mathbf{F} \cdot \mathbf{T}\,ds$
    - $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz$ where $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$
- Fundamental Theorem for Line Integrals
  - Theorem
    - C is smooth curve given by vector function $\mathbf{r}(t)$, $a \leqslant t \leqslant b$
    - f is differentiable function that ∇f is continuous on C
    - $\int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$
  - Independence of Path
    - Theorem 1
      - if and only if $\int_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C in D
      - $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D
    - Theorem 2
      - $\mathbf{F}$ is vector field continuous on open connected region D
      - if $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D
      - $\mathbf{F}$ is conservative vector field on D; there exists function f that $\nabla f = \mathbf{F}$
    - Theorem 3
      - if $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is conservative vector field
      - and P and Q have continuous first-order partial derivatives on D
      - then through D, $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$
    - Theorem 4

        - $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j}$ is vector field on open simply-connected D
        - P and Q have continuous first-order partial derivatives
        - and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ throughout D
        - then, $\mathbf{F}$ is conservative
    - Conservation of Energy
- Green's Theorem
  - Green's Theorem
    - C is positively oriented, piecewise-smooth, simple closed curve
    - D is region bounded by C
    - if P and Q have continuous partial derivatives on open region contains D,
    - $\int_C P\,dx + Q\,dy = \iint\limits_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\,dA$
  - Theorem 2
    - $A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$
- Curl and Divergence
  - Curl
    - Definition
      - $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$
      - $\text{curl}\,\mathbf{F} = \nabla \times \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$
    - Theorems
      - 1
        - if f is 3-variable function that has continuous second order partial derivatives
        - $\text{curl}(\nabla f) = 0$
      - 2
        - $\mathbf{F}$ is vector field on all $\mathbb{R}^3$ whose component functions have continuous partial derivatives
        - if $\text{curl}\,\mathbf{F} = 0$,
        - $\mathbf{F}$ is conservative vector field
  - Divergence
    - Definition
      - $\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$
    - Theorem
      - $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$
      - if P, Q, R have continuous second-order partial derivatives,
      - $\text{div}\,\text{curl}\,\mathbf{F} = 0$
  - Vector Forms of Green's Theorem
    - $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint\limits_D (\text{curl}\,\mathbf{F}) \cdot \mathbf{k}\,dA$
    - $\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \iint\limits_D \text{div}\,\mathbf{F}(x, y)\,dA$
- Parametric Surfaces and Their Areas
  - Parametric Surfaces
    - $\mathbf{r}(u, v) = x(u, v)\,\mathbf{i} + y(u, v)\,\mathbf{j} + z(u, v)\,\mathbf{k}$
    - Parametric Equations: $x = x(u, v)$, $y = y(u, v)$, $z = z(u, v)$
  - Surface of Revolution

        - $x = x$, $y = f(x)\cos\theta$, $z = f(x)\sin\theta$
        - $a \leqslant x \leqslant b$, $0 \leqslant \theta \leqslant 2\pi$
    - Tangent Planes
        - $\mathbf{r}(u,v) = x(u,v)\ \mathbf{i} + y(u,v)\ \mathbf{j} + z(u,v)\ \mathbf{k}$
        - if $\mathbf{r}_u \times \mathbf{r}_v \neq 0$, S is smooth
        - for smooth surface, $\mathbf{r}_u \times \mathbf{r}_v$ is normal vector to tangent plane
    - Surface Area
        - S: $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$, $(u,v) \in D$
        - $A(S) = \iint\limits_D |\mathbf{r}_u \times \mathbf{r}_v|\ dA$
    - Surface Area of Graph of Function
        - $x = x$, $y = y$, $z = f(x,y)$
        - $A(S) = \iint\limits_D \sqrt{1 + (\frac{\partial z}{\partial x})^2 + \left(\frac{\partial z}{\partial y}\right)^2}\ dA$
- Surface Integrals
    - Definition
        - $\iint\limits_S f(x,y,z)\ dS = \lim\limits_{m,n\to\infty} \sum\limits_{i=1}^{\infty}\sum\limits_{j=1}^{\infty} f(P_{ij}^*)\Delta S_{ij} = \iint\limits_D f(\mathbf{r}(u,v))|\mathbf{r}_u \times \mathbf{r}_v|\ dA$
    - Graphs of Functions
        - $x = x$, $y = y$, $z = g(x,y)$
        - $\iint\limits_S f(x,y,z)\ dS = \iint\limits_D f(x,y,g(x,y))\sqrt{(\frac{\partial z}{\partial x})^2 + (\frac{\partial z}{\partial y})^2 + 1}\ dA$
    - Oriented Surfaces
        - Unit Normal Vector: $\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$
    - Surface Integrals of Vector Fields
        - $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_S \mathbf{F}\cdot\mathbf{n}\ dS$
        - $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \mathbf{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\ dA$
        - $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iint\limits_D \left(-P\frac{\partial g}{\partial x} - Q\frac{\partial g}{\partial y} + R\right)\ dA$
- Stokes' Theorem
    - 1: $\int_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_S \text{curl}\ \mathbf{F}\cdot d\mathbf{S}$
    - 2
        - S_1 and S_2 are oriented surfaces with same oriented boundary curve C and both satisty Stokes' Theorem,
        - $\iint\limits_{S_1} \text{curl}\ \mathbf{F}\cdot d\mathbf{S} = \int_C \mathbf{F}\cdot d\mathbf{r} = \iint\limits_{S_2} \text{curl}\ \mathbf{F}\cdot d\mathbf{S}$
- Divergence Theorem
    - $\iint\limits_S \mathbf{F}\cdot d\mathbf{S} = \iiint\limits_E \text{div}\ \mathbf{F}\ dV$