AP Calculus BC

Version 1.5

© Ruixuan Tu (turx2003@gmail.com, University of Wisconsin-Madison)

Basics

- Perpendicular Line: $m_n \cdot m_t = -1$
- Logarithmic Functions
  - $\log_a b + \log_a c = \log_a (b \cdot c)$
  - $\log_a b - \log_a c = \log_a (\frac{b}{c})$
  - $\log_{a^n} b^m = \frac{m}{n} \log_a b$
  - $\log_a b = \frac{\log_c b}{\log_c a}$
  - $\log_a b \cdot \log_b a = 1$
- Trigonometric Functions
  - Definitions
    - $\sec\theta = \frac{1}{\cos\theta}$
    - $\csc\theta = \frac{1}{\sin\theta}$
    - $\tan\theta = \frac{\sin\theta}{\cos\theta}$
  - Formulas
    - $\sin(\theta + 2\pi k) = \sin\theta$
    - $\cos(\theta + 2\pi k) = \cos\theta$
    - $\sin(2x) = 2\sin x \cos x$
    - $\cos(2x) = \cos^2 x - \sin^2 x$
    - $\sin^2\theta + \cos^2\theta = 1$
    - $\sin(\frac{\pi}{2} + x) = \sin x$
    - $\cos(\frac{\pi}{2} + x) = -\cos x$
    - For 30° and 60°, just remember $1 : 2 : \sqrt{3}$
  - Table

θ	0	$\frac{\pi}{6}(30^\circ)$	$\frac{\pi}{4}(45^\circ)$	$\frac{\pi}{3}(60^\circ)$	$\frac{\pi}{2}(90^\circ)$	$\pi(180^\circ)$
$\sin\theta$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1
$\tan\theta$	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$	not exist	0

- Exponential Functions

x	0	1	e	10
e^x	1	e	-	-
$\ln(x)$	undef $(-\infty)$	0	1	-
$\log_{10}(x)$	undef $(-\infty)$	0	-	1

Derivatives

- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$

- $f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$
- $(f(x) \pm g(x))' = f'(x) \pm g'(x)$
- $(f(x) \cdot g(x))' = f'(x)g(x) + f(x)g'(x)$
- $(\frac{f}{g})' = \frac{f'g-fg'}{g^2}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\tan x)' = \sec^2 x$
- $(\cot x)' = -\csc^2 x$
- $(\sec x)' = \sec x \tan x$
- $(\csc x)' = -\csc x \cot x$
- $(\sin^{-1} x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\cos^{-1} x)' = -\frac{1}{\sqrt{1-x^2}}$
- $(\tan^{-1} x)' = \frac{1}{1+x^2}$
- $(\cot^{-1} x)' = -\frac{1}{1+x^2}$
- $(\sec^{-1} x)' = \frac{1}{|x|\sqrt{x^2-1}}$
- $(\csc^{-1} x)' = -\frac{1}{|x|\sqrt{x^2-1}}$
- $(e^x)' = e^x$
- $(a^x)' = a^x \ln a$
- $(\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \frac{1}{x \ln a}$
- $(x^x)' = x^x(\ln x + 1)$
- $(uvw)' = u'vw + uv'w + uvw'$
- $(y^{-1})'|_{x=f(a)} = \frac{1}{y'|_{x=a}}$ or $g'(y) = \frac{1}{f'(x)}(g = f^{-1},\ \ (x,y)$: a point on the graph)

Integration

- Properties of Indefinite Integrals
  - $\int kf(x)dx = k\int f(x)dx$ for any constant k
  - $\int (f(x) \pm g(x))dx = \int f(x)dx \pm \int g(x)dx$
- Power Formulas
  - $\int u^n du = \frac{u^{n+1}}{n+1} + C$ when $n \neq -1$
  - $\int u^{-1} du = \frac{1}{u}du = \ln|u| + C$
- Trigonometric Formulas
  - $\int \cos u du = \sin u + C$
  - $\int \sin u du = -\cos u + C$
  - $\int \sec^2 u du = \tan u + C$
  - $\int \csc^2 u du = -\cot u + C$
  - $\int \sec u \tan u du = \sec u + C$
  - $\int \csc u \cot u du = -\csc u + C$
- Exponential and Logarithmic Formulas
  - $\int e^u du = e^u + C$
  - $\int a^u du = \frac{a^u}{\ln u} + C$
  - $\int \ln u du = u \ln u - u + C$
  - $\int \log_a u du = \int \frac{\ln u}{\ln a} du = \frac{u \ln u - u}{\ln a} + C$

- By Substitution
  - $\int f(g(x))g'(x)dx = \int f(u)du$
  - Example
    - $\int x^2\sqrt{5+2x^3}dx = \int \sqrt{5+2x^3}x^2dx = \int \sqrt{5+2x^3}d(\frac{1}{3}x^3) = \int \sqrt{5+2x^3}d(\frac{1}{6}(2x^3)) = \frac{1}{6}\int \sqrt{5+2x^3}d(2x^3+5)$
    - $\frac{1}{6}\int \sqrt{5+2x^3}d(2x^3+5) = \frac{1}{6}\int u^{\frac{1}{2}}du$
    - $\frac{1}{6}\int u^{\frac{1}{2}}du = \frac{1}{6}(\frac{1}{\frac{1}{2}+1}u^{\frac{1}{2}+1}) + C \Rightarrow \frac{1}{6}\cdot\frac{2}{3}u^{\frac{3}{2}} + C = \frac{1}{9}(2x^3+5)^{\frac{3}{2}} + C$
- By Parts
  - $\int udv = uv - \int vdu$
  - u: Logarithmic, Inverse Trigonometric, Polynomial, Exponential, Trigonometric
- Partial Fraction Decomposition: $\int \frac{D}{(x-a)(x-b)}dx = \int(\frac{A}{x-a} + \frac{B}{x-b})dx = A\ln|x-a| + B\ln|x-b| + C$
- Improper Integrals
  - $\int\limits_a^\infty f(x)dx = \lim\limits_{b\to\infty}\int\limits_a^b f(x)dx$
  - $\int\limits_{-\infty}^b f(x)dx = \lim\limits_{a\to-\infty}\int\limits_a^b f(x)dx$
  - $\int\limits_{-\infty}^\infty f(x)dx = \int\limits_{-\infty}^c f(x)dx + \int\limits_c^\infty f(x)dx$
  - convergent: have result
  - divergent: have no result

Application

- Graphing: positive from quadrant 1, negative from quadrant 4
- $f''(x)$
  - >0: concave up, local minimum
  - <0: concave down, local maximum
- Linearization: $L(x) = f(x_0) + f'(x_0)\ \ (x - x_0)$
- Trapezoidal Rule
  - $T = \frac{1}{2}(f(x_0) + f(x_1))\Delta x + \frac{1}{2}(f(x_1) + f(x_2))\Delta x + ... + \frac{1}{2}(f(x_{n-1}) + f(x_n))$
  - $T = \frac{\Delta x}{2}(f(x_0) + 2f(x_1) + 2f(x_2) + ... + 2f(x_{n-1}) + 2f(x_n)$
  - $T = \frac{LRAM_n + RRAM_n}{2}$
  - $MRAM = f(\frac{x_n + x_{n+1}}{2}) \neq T$
- Euler's Method: $\Delta y = \frac{dy}{dx} - \Delta x$
- Exponential Model
  - $\frac{dy}{dx} = ky$
  - $y = Ce^{kx}$
  - $y = y_0Ce^{kx}$ ($y = y_0$ when $x = 0$)
- Logistic Model
  - $\frac{dy}{dx} = ky(a-y)$
  - $y = \frac{a}{1+Ce^{kx}}$
- Arc Length: $L = \int\sqrt{1 + (\frac{df}{dx})^2}dx$
- Distance: $x = \int\sqrt{(\frac{dy}{dt})^2 + (\frac{dx}{dt})^2}dt$

- Volume: $V = \int S dx$
- Parametric Function
  - $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$
  - $\frac{d^2y}{dx^2} = \frac{d}{dx}(\frac{\frac{dy}{dt}}{\frac{dx}{dt}}) = \frac{\frac{d}{dt}(\frac{dy}{dx})}{\frac{dx}{dt}}$
- Polar Coordinate
  - $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$
  - $\begin{cases} r^2 = x^2 + y^2 \\ \tan\theta = \frac{y}{x} \end{cases}$
  - $\frac{dy}{dx} = \frac{d(r\sin\theta)}{d(r\cos\theta)} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta}$
  - Area: $A = \int\limits_a^b \frac{1}{2}r^2 d\theta$
  - Arc Length: $L = \int \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta$

Series

- Tests
  - Ratio Test
    - $\sum\limits_{n=0}^{\infty} a_n = \sum_{k=0}^{\infty} cr^k = c + cr + cr^2 + cr^3 + \cdots + cr^k$
    - $L = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \begin{cases} L < 1 & \text{convergent} \\ L > 1 & \text{divergent} \\ L = 1 & \text{pending / test fails} \end{cases}$
  - Integral Test
    - For $\int\limits_1^{\infty} f(x)dx$
    - If $\lim_{n\to\infty} a_n \neq 0$, $\sum\limits_{n=1}^{\infty} a_n$ divergent
    - If $\lim_{n\to\infty} a_n = 0$, $\sum\limits_{n=1}^{\infty} a_n$ convergent
  - Comparison Test
    - $S_a = \sum\limits_{n=1}^{\infty} a_n$, $S_b = \sum\limits_{n=1}^{\infty} b_n$
    - If S_a converges, $S_a > S_b$, S_b converges
    - If S_b diverges, $S_b > S_a$, S_a diverges
  - Alternating Series Test
    - Convergent if
      - $\lim_{n\to\infty} b_n = 0$
      - b_n decreasing
- Series
  - p-Series
    - $\sum\limits_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} p > 1 & \text{convergent} \\ p \leq 1 & \text{divergent} \end{cases}$
    - Harmonic Series: $p = 1$
  - Geometric Series

- $\sum_{n=1}^{\infty} ar^{n-1} \begin{cases} |r| < 1 & \text{convergent} \\ |r| \geq 1 & \text{divergent} \end{cases}$
- $|r| < 1$
  - $\sum_{n=1}^{k} ar^{n-1} = a(\frac{1-r^{k+1}}{1-r})$
  - $\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$
- $r = \frac{a_{n+1}}{a_n}$ (Ratio Test)

◦ Alternating Series
- $S_a = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots + (-1)^{n+1} a_n \ (b_n > 0)$
- If $a_n > a_{n+1}$ and $\lim_{n\to\infty} a_n = 0$, convergent
- Error Bound: the next term
- Bound Estimate Theorem: $\sum_{n=1}^{\infty} (-1)^{n+1} a_n = L, |S_n - L| \leq |a_{n+1}|$

◦ Power Series
- $x = a$: $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n$
- $\sum_{n=1}^{\infty} |a_n| \ \text{convergent} \Rightarrow \sum_{n=1}^{\infty} a_n \ \text{convergent}$
- $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n} \cdot (x-a)| = \lambda \begin{cases} 0 \leq \lambda < 1 & \text{convergent} \\ \lambda > 1 & \text{divergent} \\ \lambda = 1 & \text{pending / test fails} \end{cases}$

◦ Taylor Series
- $x = c$: $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!} (x-c)^n$
- Lagrange Error Bound
  - $f(x) = P_k(x) + R_k(x)$
  - $P_k(x) = \sum_{n=0}^{k} \frac{f^{(n)}(c)}{n!} (x-c)^n$
  - $R_k(x) = \frac{f^{(k+1)}(\xi)}{(k+1)!} (x-c)^{k+1}, \quad c < \xi < x$

◦ Maclaurin Series
- Taylor Series, $c = 0$
- $\frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad (|x| < 1)$
- $\frac{1}{1+x} = 1 - x + x^2 - \cdots = \sum_{n=0}^{\infty} (-1)^n x^n \quad (|x| < 1)$
- $e^x = 1 + x + \frac{x^2}{2!} + \cdots = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{ all real } x)$
- $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \quad (\text{ all real } x)$
- $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \quad (\text{ all real } x)$
- $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots = \sum_{x=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} \quad (-1 < x \leq 1)$
- $(1+x)^p = \sum_{k=0}^{\infty} \frac{p(p-1)\ldots(p-k+1)}{k!} x^k, |x| < 1, p \in \mathbb{R}$